Implementation of the Finite Isotropic Linear Cosserat Models based on the Weak Form

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Abstract: The Cosserat models fall into the group of the extended continuum media. The main idea of the Cosserat models was proposed by Cosserat brothers and was simplified to the linear models by Eringen et al. in the early 1960s. The Cosserat model, called also micropolar theorem, can be used to handle the microstructure problems like man-made grid structure, foam or bone and the granular materials. The Cosserat models are capable of treating the size effects (characteristic length) in a natural manner using six material moduli for the isotropic elastic cases instead of two (λ and μ) for the classical continuum mechanics. This model involves two constitutive laws corresponding to two kinds of balance equation. The first one handles the stress-strain relation and the second treats the couple stress-curvature tensor relation. It is of importance to note that neither stress tensor nor strain tensor is symmetric and the strain tensor contains the microrotation term. In this study, Comsol Multiphysics enables us to put into practice 3D Cosserat models for the multi-scale modeling.

Keywords: linear isotropic Cosserat materials, size effect, microrotation, characteristic length

1 Introduction

We investigate the numerical response of the isotropic linear Cosserat models by means of Comsol Multiphysics. Comparing to the classical linear elasticity or so called **Cauchy's medium**, there are three microrotations besides three displacements. As a result, six state variables should be taken into account in the numerical simulations. As pointed out earlier, the main advantage of the additional parameters is that they consider the size effect in an explicit manner. It is needed to be mentioned that the size effect could not be described by the classical continuum mechanics. The finite element analysis of the Cosserat models has been done by several researchers for the 2D structural mechanics problems, e.g., [1] and [2]. The finite element analysis of the nonlinear Cosserat models has been also recently achieved by means of FEAP [1] which has its specific restrictions [3]. Additionally, the elastic-plastic Cosserat models have been developed using multigrid solvers for the Boundary Value Problems (BVP) by another research team in Germany (Wieners et al.) [4]. There are several attempts to perform elastic and elastic-plastic Cosserat analyses using Abaqus [2] [5]. Due to the additional unknowns (microrotations), lack of an experimental method for measuring up these microrotations and difficulties which arise from the numerical methods such as FEM, nowadays the Cosserat models are being hardly accepted in the structural mechanics field. In spite of the fact that the Cosserat models get more complicated particularly for 3D cases, we have found out the very good numerical results comparing to ones which have been previously done by FEAP. Moreover, none of these packages (FEAP, Abaqus) is able to apply the implicit Dirichlet boundary condition like Comsol Multiphysics for the Coupled Partial Differential Equations (CPDEs). The Comsol Multiphysics readily did this job in the cur-

[1] FEAP is a general purpose finite element analysis program which is designed for research and educational use. Source code of the full program is available for compilation using Windows (Compaq or Intel compiler), LINUX or UNIX operating systems, and Mac OS X based Apple systems. Extracted from the following link: "http://www.ce.berkeley.edu/ rlt/feap/"

[2] Abaqus is a commercial software package for finite element analysis.

rent paper. This comes from the fact that Comsol Multiphysics is extremely flexible for the Coupled Partial Differential Equations (CPDEs) system. It is important to note that the BVP-based programmes are also capable of treating this job [4]. Considering the same capacity of scientific computation, Comsol Multiphysics speeds up the computations because of its parallelized direct solvers and it manages much better the shared memory systems to do the individual digital tasks. To our knowledge, the isotropic linear Cosserat models for 3D case are programmed for the first time in Comsol Multiphysics in this study. The authors have also successfully computed isotropic nonlinear 3D Cosserat models using Comsol Multiphysics (Some comparisons have been done to evaluate the numerical experiments [3]). In the next section, we scrutinize the strong form of linear isotrpic Cosserat models.

2 The linear elastic Cosserat model in the variational form and strong form

For the **displacement** u and the **skew-symmetric infinitesimal microrotation** $\overline{A}$ we consider the **two-field** minimization problem:

$$I(u,\overline{A}) = \int_\Omega W_{\mathrm{mp}}(\overline{\varepsilon}) + W_{\mathrm{curv}}(\nabla\,\mathrm{axl}(\overline{A})) - < f, u > dx \mapsto \min .\ \text{w.r.t. } (u,\overline{A}). \tag{1}$$

under the following constitutive requirements and boundary conditions

$$\overline{\varepsilon} = \nabla u - \overline{A},\ \phi := \mathrm{axl}(\overline{A}), \quad u_{|\Gamma} = u_{\mathrm{d}}. \tag{2}$$

$\overline{\varepsilon}$ is **First Cosserat stretch tensor** and $u_{|\Gamma}$ is **displacement boundary conditions**. The strain energy is noted as:

$$W_{\mathrm{mp}}(\overline{\varepsilon}) = \mu\,\|\mathrm{sym}\,\overline{\varepsilon}\|^2 + \mu_c\,\|\mathrm{skew}\,\overline{\varepsilon}\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\,[\mathrm{sym}\,\overline{\varepsilon}]^2. \tag{3}$$

and the curvature energy is

$$W_{\mathrm{curv}}(\nabla\phi) = \frac{\gamma+\beta}{2}\,\|\mathrm{sym}\,\nabla\phi\|^2 + \frac{\gamma-\beta}{2}\,\|\mathrm{skew}\,\nabla\phi\|^2 + \frac{\alpha}{2}\,\mathrm{tr}\,[\nabla\phi]^2. \tag{4}$$

The strain energy W_{mp} and the curvature energy W_{curv} are general isotropic quadratic functions for the **infinitesimal non-symmetric first Cosserat strain tensor** and the **micropolar curvature tensor**. The parameters μ and λ are the classical Lamé's moduli and α, β, γ are further micropolar moduli in [N.m]. The additional parameter μ_c in the strain energy density term is the **Cosserat couple modulus**. For $\mu_c = 0$ the two fields of displacement u and $\overline{A}$ are decoupled and one is left formally with classical linear elasticity for the displacement.The strong form can be obtained from the weak form of this model and it is presented for the balance of linear momentum as below:

$$\mathrm{Div}\,\sigma = f, \quad \sigma = 2\mu\cdot\mathrm{sym}\,\overline{\varepsilon} + 2\mu_c\cdot\mathrm{skew}\,\overline{\varepsilon} + \lambda\cdot\mathrm{tr}\,[\overline{\varepsilon}]\cdot 1\!\!1 \tag{5}$$

and for the balance of angular momentum :

$$\begin{aligned} -\,\mathrm{Div}\,m &= 4\,\mu_c\cdot\mathrm{axl}(\mathrm{skew}\,\overline{\varepsilon}),\\ &= 2\cdot\mathrm{axl}(\mathrm{skew}\,\sigma)\\ m &= \gamma\,\nabla\phi + \beta\,\nabla\phi^T + \alpha\ \mathrm{tr}\,[\nabla\phi]\cdot 1\!\!1\\ \phi &= \mathrm{axl}(\overline{A}), u_{|\Gamma} = u_d \end{aligned} \tag{6}$$

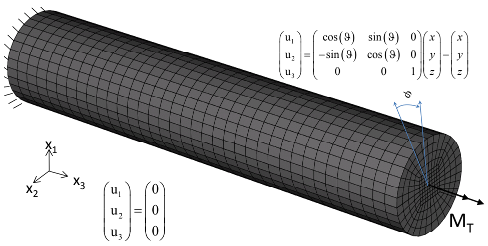
We run this Cosserat model based on the curvature energy using the pointwise assumption ($\frac{\mu L_c^2}{2}\|\nabla\phi\|^2$). It is appropriate to make the first choice among the available assumptions for simplicity (there are three different methods, i.e., pointwise case, symmetric case and conformal case [6]). The other cases have been separately treated in another study using Comsol Multiphysics and the results display and confirm the boundedness problem for the smaller specimens as expected before [6]. This choice leads to $\alpha = 0$, $\beta = 0$, $\gamma = \mu L_c^2$ and then the couple stress expression can be written at once $m = \mu L_c^2 \nabla\phi$ [3]. As pointed out previously, in this paper, we solely concentrate

[3] For the symmetric case, α is still equal to zero but $\beta = \gamma = \frac{1}{2}\mu L_c^2$ ($m = \frac{1}{2}\mu L_c^2\nabla\phi + \frac{1}{2}\mu L_c^2\nabla\phi^T$). The conformal case [6] gets nonzero value for α ($\alpha = -\frac{1}{3}\mu L_c^2$ and $\beta = \gamma = \frac{1}{2}\mu L_c^2$) and we obtain the most complete form ($m = -\frac{1}{3}\mu L_c^2\,\mathrm{tr}[\nabla\phi] + \frac{1}{2}\mu L_c^2\nabla\phi + \frac{1}{2}\mu L_c^2\nabla\phi^T$).

on the pointwise case and we solved the CPDEs of isotropic linear Cosserat model for a clamped circular bar under pure torsion (one end is fixed and another is under an applied torque).

3 Finite element analysis

3.1 preliminaries



Number of elements	Young's modulus (MPa)	Type of elements	Poisson's ratio	DOFs
6912	1E6	Hexahedral	0.3	198 618

Figure 1: The geometry, data of the torsion problem and Mesh density illustration for considered circular bar.

All our computations have been achieved by means of a user-written code within the Comsol Multyphysics based on the weak forms solution. According to discussed balance equation (Eq.5 and Eq.6) there are six available state variables (three for the displacement using quadratic Lagrange shape function and three for microrotation using linear Lagrange shape function) whose computations need to be done via the linear Coupled Partial Differential system of equations using the momentum and angular momentum balance equations. For our numerical computation, we consider a circular bar (Diameter =2mm, Height=10mm) submitted to the torsion angle θ at the end (Fig.1).

3.2 Numerical results verification by the analytical solutions for pure torsion

If the sample has circular section in the torsion problem an analytical solution for the linear Cauchy-elastic problem is available which connects the rotation at the upper face with the applied angle. For the numerical computations, we applied a very small exact angles $\theta = 2^{\circ}$ based on the Dirichelt boundary condition, because in this case we are as close as possible to the first phrase of the Taylor series expansion for trigonometric functions. The comparison between the analytical solution and FEM solution using 50000 DOFs and 200000 DOFs for a displacement gradient components $u_{1,2}$ versus distance along longitudinal direction of the cylindrical bar is presented in Fig. 2:

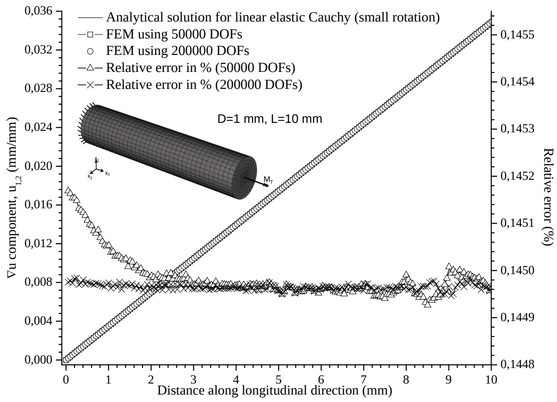


Figure 2: The relative error between the analytical solution and FEM-Comsol Multiphysics solution for 50000 DOFs and 200000 DOFs on the cylindrical bar rotated $\theta = 2^{\circ}$ at the top and fixed at the bottom of bar.

For given applied rotation angle θ at the upper face we have:

$$M_T^{\text{exact}} = \int_{\partial\Omega^+} (x\,\sigma_{32}^{\text{exact}} - y\,\sigma_{31}^{\text{exact}})\,\mathrm{d}x\,\mathrm{d}y\,. \tag{7}$$

here $\sigma_{ij}^{\text{exact}}$ is obtained exact Cauchy stress defined as bellow:

$$\sigma^{\text{exact}}(F) = \frac{1}{\det F}\, S_1(F)\, F^T\,. \tag{8}$$

here S_1 is the first Piola Kirchhoff stress tensor in nonlinear elasticity, $F = \nabla\varphi = 1\!\!1 + \nabla u$ is the deformation gradient and ∇u or $\nabla \otimes u$ is called the displacement gradient in the modern structural mechanics. It is very important to remind that no artificial boundary condition for microrotations was considered in the prepared Comsol Multiphysics models.

3.3 Limit case: Cauchy elasticity in the linear Cosserat elasticity

It is easy to see that one obtains the linear elasticity displacement solution for $\mu_c = \mu$

and $L_c = 0$. Since the curvature energy is absent, the balance of angular momentum equation reduces to the skew $\nabla u = \overline{A}$. Thus the skew-symmetric parts in the balance of force equation cancel and the displacement u is again the Cauchy displacement. A second alternative case is to take $\mu_c = 0$ and L_c=Large. In this situation, the system of equation decouples. For very large L_c, the microrotations approach a constant value over the entire body if there are no boundary condition imposed on the microrotations. In this work, we evaluated these two limit cases by considering $L_c = 10E6$ [mm] for the second limit case and it is presented in Fig.3 against the Cauchy solution previously calculated in the text.

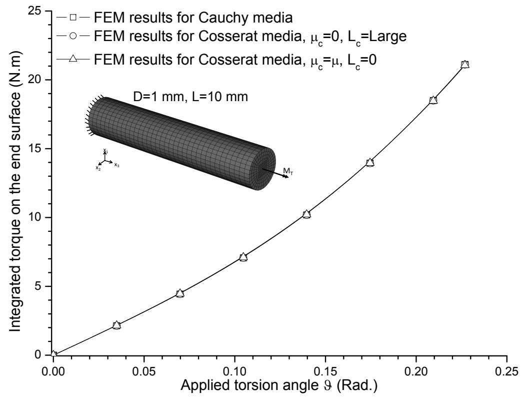


Figure 3: The comparison among Cauchy media and two limit cases in Cosserat linear elasticity.

We used only 198618 DOFs to obtain 0.14 percent of error. Our numerical results substantiate that our implementation for the Cosserat model perfectly matches the linear elastic solution.

4 Linear Cosserat elasticity implementation in Comsol Multiphysics

We apply the exact angle θ, exact rotation by the Dirichlet boundary condition (Eq.9), at the top from 0 to $\frac{13\pi}{180}$(rad) for each value of L_c which is varied from zero to 10E6 mm. The values of Cosserat couple modulus has been considered to be equal to μ ($\mu_c = \mu = \frac{E}{2(1+\nu)}$). This make it possible to apply large rotation angles without any approximation. Hence, the finite rotation computation is quite feasible by this type of boundary condition. It is worth mentioning that the microrotations are always considered as a displacement-independent parameter in the Cosserat theory, due to this fact, no artificial boundary condition has been deemed for the microrotations and they are computed implicitly by the applied CPDEs. The computations have been provided by "**General Weak Form**" option of "Comsol Multiphysics 3.4 Hotfix 2".

$$\begin{aligned} u_1 &= 0,\ u_2 = 0,\ u_3 = 0, \text{ at the bottom} \\ u_1 &= x\cos\theta + y\sin\theta - x, \\ u_2 &= -x\sin\theta + y\cos\theta - y, \\ u_3 &= 0, \text{ at the top} \end{aligned} \qquad (9)$$

The numerical solution for linear Cosserat elasticity with pointwise positive curvature exhibits more stiffness for higher values of L_c in an asymptotic manner.

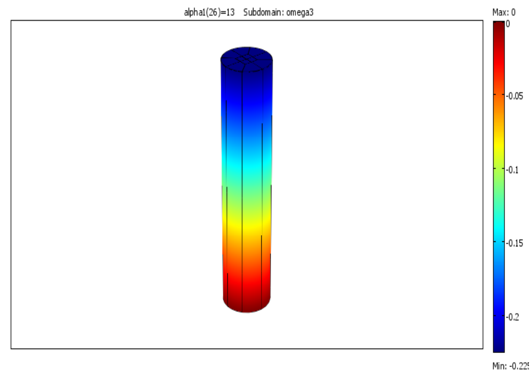


Figure 1: Macro-rotation in longitudinal direction for different L_c values L_c=0.

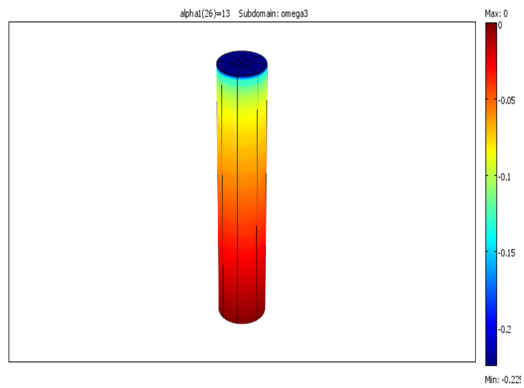


Figure 2: Macro-rotation in longitudinal direction for different L_c values L_c=5 mm.

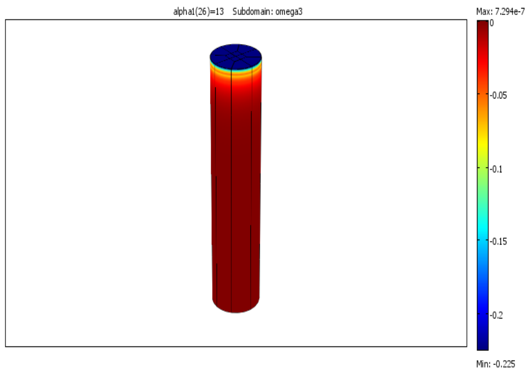


Figure 3: Macro-rotation in longitudinal direction for different L_c values L_c=10E6 mm.

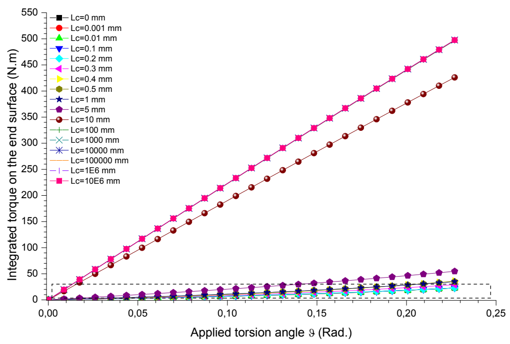


Figure 4: Size effect appearance according to the L_c value.

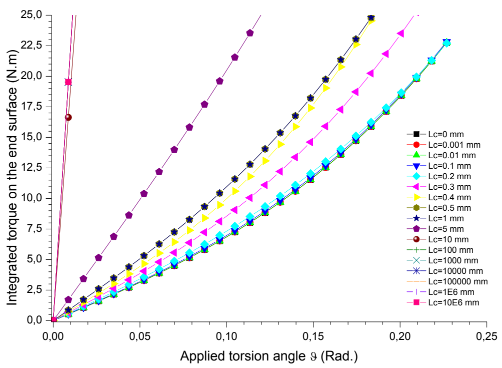


Figure 5: Extended torque value between 0 to 25 N.m.

4.1 Size effect-Log diagram

According to the last results, it is possible to plot the torque magnitude at the top of cylindrical bar versus "$Log(L_c)$" for a given torsion angle θ. Evidently, we find the upper and lower bound for the stiffness M_T. In the diagram, we distinguish three specific zones : **Zone I** rends toward linear Cauchy elasticity with no size effects, **Zone II** is an intermediate zone in which the size effect appears and we can clearly distinguish the Cosserat effects for the numerical models, **Zone III** describes a situation where the microrotation nearly constant with the limit behavior [6].

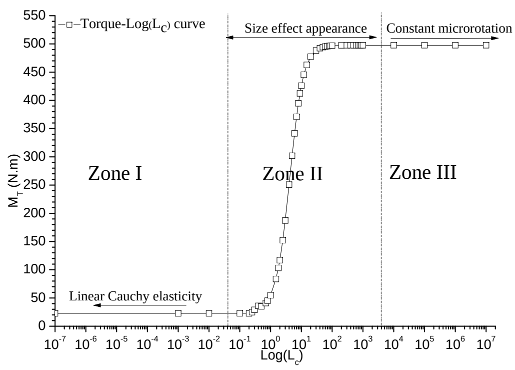


Figure 6: Torque versus L_c in a semi-logarithmic diagram for the cylindrical bar.

5 Conclusions and perspectives

The isotropic linear Cosserat equations have been successfully solved using Comsol Multiphysics 3.4 in this paper. The relevant choice for the Lagrange shape functions (quadratic Lagrange shape function and linear Lagrange shape function for the displacements and microrotations) makes it possible to get the mesh-independent solutions. In accordance with the huge number of computations [4] for different shape functions, the authors conclude that this Anszat shape function is the best choice for Cosseral models [3, 6]. As far as the authors knowledge and some attempts, the other FEM software packages are not as efficient as Comsol Multiphysics to apply the

[4] The authors used a very modest PC configuration. To reduce the computation runtime the parallelized direct solver "PARDISO" is used and it is recognized as the best and the most efficient solver for both linear and nonlinear Cosserat models. Here an Intel-System with 8 GB memory, dual core 3.2 GHz processor and 83 GB hard disk has been used using Linux64 (Commercial-free Linux "UBUNBTU"). This Operating System enables us to use easily the dedicated swap memory for very large problems. A computation with 550000 DOFs for the linear Cosserat would last about 4 days!!

geometrically exact Cosserat model same as those applied in this study.

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Appendix

Let $\Omega \subset R^3$ be a bounded domain with Lipschitz boundary.$\partial\Omega$ and let Γ be a smooth subset of $\partial\Omega$ with non-vanishing 2-dimensional Hausdorff measure. For $a, b \in R^3$ we let $\langle a, b\rangle_{R^3}$ denote the scalar product on R^3 with associated vector norm $\|a\|^2_{R^3} = \langle a, a\rangle_{R^3}$. We denote by $\mathrm{M}^{3\times 3}$ the set of real 3×3 second order tensors, written with capital letters and sym denotes symmetric second orders tensors. The standard Euclidean scalar product on $\mathrm{M}^{3\times 3}$ is given by $\langle X, Y\rangle_{\mathrm{M}^{3\times 3}} = \mathrm{tr}\, XY^T$, and thus the Frobenius tensor norm is $\|X\|^2 = \langle X, X\rangle_{\mathrm{M}^{3\times 3}}$. In the following we omit the index $R^3, \mathrm{M}^{3\times 3}$. The identity tensor on $\mathrm{M}^{3\times 3}$ will be denoted by $1\!\!1$, so that $\mathrm{tr}\, X = \langle X, 1\!\!1\rangle$. We set $\mathrm{sym}(X) = \frac{1}{2}(X^T + X)$ and $\mathrm{skew}(X) = \frac{1}{2}(X - X^T)$ such that $X = \mathrm{sym}(X) + \mathrm{skew}(X)$. For $X \in \mathrm{M}^{3\times 3}$ we set for the deviatoric part $\mathrm{dev}\, X = X - \frac{1}{3}\ \mathrm{tr}\, X\, 1\!\!1 \in \mathfrak{sl}(3)$ where $\mathfrak{sl}(3)$ is the Lie-algebra of traceless matrices. The set $\mathrm{Sym}(n)$ denotes all symmetric $n \times n$-matrices.

The Lie-algebra of $\mathrm{SO}(3) := \{X \in \mathrm{GL}(3)\ |X^T X = 1\!\!1,\ \det X = 1\}$ is given by the set $\mathfrak{so}(3) := \{X \in \mathrm{M}^{3\times 3}\ \ |X^T = -X\}$ of all skew symmetric tensors. The canonical identification of $\mathfrak{so}(3)$ and R^3 is denoted by $\mathrm{axl}\,\overline{A} \in \mathrm{R}^3$ for $\overline{A} \in \mathfrak{so}(3)$. Note that $(\mathrm{axl}\,\overline{A}) \times \xi = \overline{A}.\xi$ for all $\xi \in \mathrm{R}^3$, such that

$$\mathrm{axl}\begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix} := \begin{pmatrix} -\gamma \\ \beta \\ -\alpha \end{pmatrix} \tag{1}$$

$$\begin{aligned} \overline{A}_{ij} &= \sum_{k=1}^{3} -\epsilon_{ijk} \cdot \mathrm{axl}\,\overline{A}_k\,, \\ \|\overline{A}\|^2_{\mathrm{M}^{3\times 3}} &= 2\ \|\mathrm{axl}\,\overline{A}\|^2_{\mathrm{R}^3} \\ \langle \overline{A}, \overline{B}\rangle_{\mathrm{M}^{3\times 3}} &= 2\langle \mathrm{axl}\,\overline{A}, \mathrm{axl}\,\overline{B}\rangle_{\mathrm{R}^3} \end{aligned} \tag{2}$$

where ϵ_{ijk} is the totally antisymmetric permutation tensor. Here, $\overline{A}.\xi$ denotes the application of the matrix $\overline{A}$ to the vector ξ and $a \times b$ is the usual cross-product. Moreover, the inverse of axl is denoted by anti and defined by

$$\begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix} := \mathrm{anti}\begin{pmatrix} -\gamma \\ \beta \\ -\alpha \end{pmatrix} \tag{3}$$

$$\mathrm{axl}(\mathrm{skew}(a \otimes b)) = -\frac{1}{2}\, a \times b \tag{4}$$