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Numerical Calculation of Effective Density and Compressibility Tensors in Periodic Porous Media: A Multi-Scale Asymptotic Method

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Characterization and prediction of the acoustic properties of absorbing porous materials

- Determination of effective density and compressibility tensors
  - on the analytical procedures
    - classical model: Kirchhoff (1868)
    - standard model; Johnson *et al.* (1987), Champoux and Allard (1991), Allard(1993), Lafarge *et al.* (1997)
    - general model: Pride *et al.*(1993), Lafarge (1993)
  - on the numerical procedures
    - collocation method: Chapman and Higdon (1992)
    - boundary element method: Borne (1992)
    - finite element method: Zhou and Sheng (1989), Gasser (2003), Gasser *et al.* (2005), Perrot *et al.* (2008)
- None of the methods offer satisfactorily the possibility of extending the evaluation to general geometries and materials

Introduce a consistent and general approach

- Review the multi-scale asymptotic method (MAM)
  - decoupled set of frequency-domain boundary value problems for micro-scale visco-thermal response
  - relationship between the macro-scale material description and the micro-scale structure information
- Employ COMSOL with periodic boundary conditions on a 3D unit fluid cell (UFC)
  - frequency-dependent effective density and compressibility tensors
  - acoustic absorption properties of porous materials

Frequency-based visco-thermal problem for rigid porous media

- In the visco-thermal fluid flow

$$\frac{p}{P_0} = \frac{\rho}{\rho_0} + \frac{\tau}{T_0}$$

$$\rho_0 i\omega \mathbf{u} = -\nabla p + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu \Delta \mathbf{u}$$

$$i\omega \frac{\rho}{\rho_0} = -\nabla \cdot \mathbf{u}$$

$$\rho_0 i\omega C_p \tau = i\omega p + K\Delta\tau$$

- On the fluid-solid interface

$$\mathbf{u} = 0 \quad \tau = 0$$

Two key assumptions

- Existence of the small parameter

$$\epsilon = h/L \ll 1 \Rightarrow y = \epsilon^{-1} x$$

  where
  - h and L $(= \lambda/2\pi)$ are the characteristic lengths at the micro-scopic and macro-scopic level, respectively
  - x and y are the macro- and micro-variations, respectively
- Periodicity of the microstructure

Introduce expansions

$$\mathbf{u} = \mathbf{u}^0(x,y) + \varepsilon \mathbf{u}^1(x,y) + \varepsilon^2 \mathbf{u}^2(x,y) + \cdots$$

$$p = p^0(x,y) + \varepsilon p^1(x,y) + \varepsilon^2 p^2(x,y) + \cdots$$

$$\tau = \tau^0(x,y) + \varepsilon \tau^1(x,y) + \varepsilon^2 \tau^2(x,y) + \cdots$$

and

$$\nabla = \nabla_x + \frac{1}{\varepsilon}\nabla_y \quad \Delta = \Delta_x + \frac{2}{\varepsilon}\Delta_{xy} + \frac{1}{\varepsilon^2}\Delta_y$$

Three dimensional relationships: Boutin *et al.* (1998)

- Viscosity behavior at the micro-scopic level

$$Q_L = |\nabla p|/|\mu \Delta \mathbf{u}| \approx O(\epsilon^{-2}) \quad RT_L = |\rho_0 i\omega \mathbf{u}|/|\mu \Delta \mathbf{u}| \approx O(\epsilon^{-2})$$

- Thermal exchange at the pore scale

$$N_L = |\rho_0 i\omega C_p \tau|/|K \Delta \tau| \approx O(\epsilon^{-2})$$

Updated momentum balance and thermal transfer equations

$$\rho_0 i\omega \mathbf{u} = -\nabla p + \varepsilon^2 \left[(\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) + \mu \Delta \mathbf{u}\right]$$

and

$$\rho_0 i\omega C_p \tau = i\omega p + \varepsilon^2 \left[K \Delta \tau\right]$$

Following the methodology introduced by Lafarge *et al.*(1997),

- Linear relationships between velocity ($\mathbf{u}^0$) and pressure (p^0) and pressure gradient ($\nabla_x p^0$)

$$\mathbf{u}^0(x,y) = -\frac{\mathbf{k}(y,\omega)}{\mu} \cdot \nabla_x p^0(x)$$

$$p^1(x,y) = -\alpha(y,\omega) \cdot \nabla_x p^0(x) + \hat{p}^1(x)$$

- Linear relationship between temperature (τ) and time-derivative pressure ($i\omega p^0$)

$$\tau^0(x,y) = \frac{k'(y,\omega)}{K} i\omega p^0(x)$$

Decoupled set of micro-scale boundary value problems suitable for incorporation into COMSOL

- Momentum equation with no-slip boundary condition
  - in Ω_f

$$i\omega \frac{\rho_0}{\mu}\mathbf{k} + \nabla\alpha \cdot \mathbf{I} - \Delta\mathbf{k} = \mathbf{I}$$

$$\nabla \cdot \mathbf{k} = \mathbf{0}$$

  - on Γ

$$\mathbf{k} = \mathbf{0}$$

where

- Ω : unit fluid cell volume
- Ω_f : fluid-filled pore volume
- Γ : fluid-solid interface
- $\mathbf{I}$: 3x3 identity matrix

Decoupled set of micro-scale boundary value problems suitable for incorporation into COMSOL (cont.)

- Thermal transfer equation with isothermal boundary condition
  - in Ω_f

$$\frac{\mathrm{Pr}}{\mu} i\omega\rho_0 k' - \Delta k' = 1$$

  - on Γ

$$k' = 0$$

  where $\mathrm{Pr} = \mu C_p / K$ denotes the Prandtl number
- Periodic conditions on Ω

$$\mathbf{k}, \quad \alpha, \quad k'$$

At a given frequency, estimate macro-scale effective density ($\rho_{\mathbf{eff}}$) and compressibility (χ_{eff}) tensors as follows:

$$i\omega\rho_{\mathbf{eff}} \cdot \left\langle \mathbf{u}^0 \right\rangle = -\nabla_x p^0 \quad i\omega\chi_{\text{eff}} p^0 = -\nabla_x \cdot \left\langle \mathbf{u}^0 \right\rangle$$

with

$$\rho_{\mathbf{eff}} = \frac{\mu\phi}{i\omega}\hat{\mathbf{k}}^{-1} \quad \text{and} \quad \chi_{eff} = \frac{1}{\gamma P_0}\left[\gamma - (\gamma - 1)\frac{\rho_0 \,\mathrm{Pr}}{\mu}\frac{i\omega\hat{k}'}{\varphi}\right]$$

where

- γ : specific heat ratio
- $\langle \bullet \rangle = \int_\Omega \bullet d\Omega/\Omega_f$: fluid-phase average
- $\phi = \Omega_f/\Omega$: porosity
- $\hat{\mathbf{k}} = \phi \langle \mathbf{k} \rangle, \hat{k}' = \phi \langle k' \rangle$: dynamic viscous and thermal permeability tensors

For the validation procedure, chose the rigid porous medium made of FCC structure introduced by Gasser (2003) and Gasser *et al.* (2005)

- Subject to mirror symmetry (y- and z-directions) and translational periodicity (x-direction)
- Restrict the pressure gradient only to the x-axis
- Provide the following material and geometric properties:

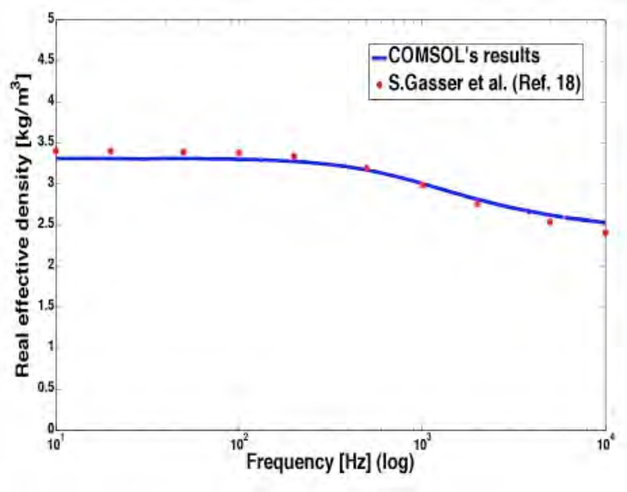
$$\rho_0 = 1.293\mathrm{kg/m^3} \quad \mathrm{T_0} = 300\mathrm{K} \quad \mathrm{P_0} = 10^5\mathrm{Pa} \quad \mathrm{Pr} = 0.715$$

and

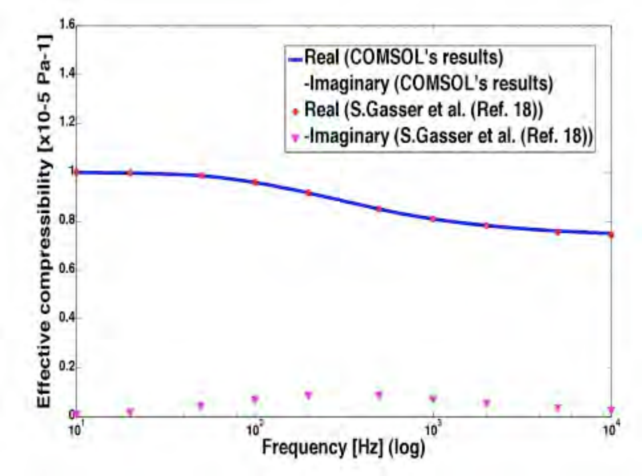
$$\mu = 1.72 \times 10^{-5}\mathrm{kg/ms^{-1}} \quad \gamma = 1.4 \quad \mathrm{R} = 1\mathrm{mm} \quad \mathrm{r} = 150\mu\mathrm{m}$$

Original FCC structure vs Meshed FCC UFC

Effective density via frequency (10 to 10,000 Hz)



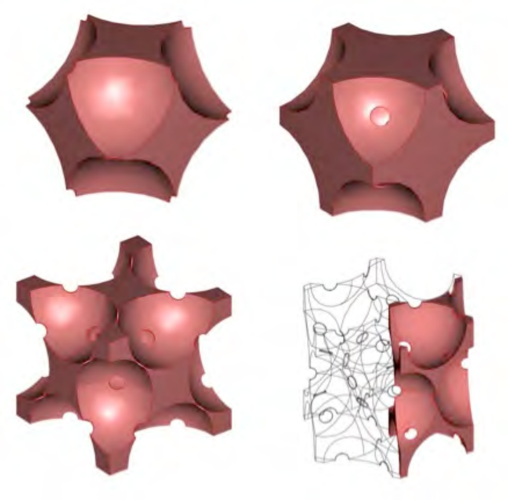
Effective compressibility via frequency (10 to 10,000 Hz)



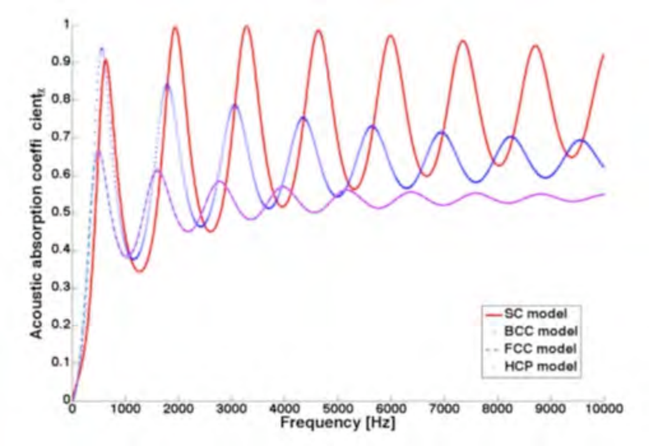
Acoustic properties of a FCC sphere stacking vs those available in the literature

	Chapman and Higdon	Gasser	COMSOL
ϕ	0.26	0.26	0.26
$\hat{k}_0$	6.95×10^{-10}	6.83×10^{-10}	6.73×10^{-10}
$\hat{\alpha}_0$	NA	2.63	2.24
$\hat{\alpha}_\infty$	1.61	1.66	1.65
$\hat{k}'_0$	NA	0.274×10^{-10}	0.272×10^{-10}
$\hat{\alpha}'_0$	NA	1.85	1.87
Λ	0.124×10^{-3}	0.164×10^{-3}	0.173×10^{-3}
Λ'	NA	0.249×10^{-3}	0.247×10^{-3}

3D classical unit fluid cell geometries (SC, BCC, FCC, HCP)



Acoustic absorption coefficients via frequency (10 to 10,000 Hz)



Concluding remarks consistent and general approach for numerically computing frequency-dependent effective density and compressibility tensors for periodic porous materials

- Formulation suitable for incorporation into COMSOL to avoid the complication and the limitation of working with, unlike previous investigations
- Present approach used to analyze the acoustic absorption properties of several packing geometries at the micro-scale level