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Solving Distributed Optimal Control Problems for the Unsteady Burgers Equation in COMSOL Multiphysics

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Abstract: We use COMSOL Multiphysics for solving distributed optimal control of unsteady Burgers equation without constraints and with pointwise control constraints. Using the first order optimality conditions, we apply projection and semi-smooth Newton methods for solving the optimality system. We have applied the standard approach by integrating the state equation forward in time and the adjoint equation backward in time using gradient method. We also consider the optimality system in the space-time cylinder as an elliptic equation and solve it adaptively. Numerical results computed by the gradient method, adaptive and non-adaptive elliptic solvers of COMSOL Multiphysics are presented for both the unconstrained and the control constrained cases.

Keywords: Optimal control, finite elements, Burgers equation, COMSOL Multiphysics.

1 Introduction

Burgers equation plays an important role in fluid dynamics as a first approximation to complex diffusion convection phenomena. It was used as a simplified model for turbulence and in shock waves. Analysis and numerical approximation of optimal control problems for Burgers equation are important for the development of numerical methods for optimal control of more complicated models in fluid dynamics like Navier-Stokes equations.

Recently, several papers appeared dealing with the optimal control of Burgers equation. A detailed analysis of distributed and boundary control of stationary and unsteady Burgers equation and the approximation of the optimality system with augmented Lagrangian SQP (sequential quadratic programming) method are given in [9]. In [7], the SQP, primal-dual active set and semi-smooth Newton methods are compared for distributed control problems related with the stationary Burgers equation with pointwise control constraints. Distributed control problems for unsteady Burgers equation with and without control constraints are investigated numerically using SQP methods in [1, 8, 10]. Different time integration methods like implicit Euler and Crank-Nicholson methods were considered for solving the adjoint equations arising by optimal control of unsteady Burger equation in [3]. In contrast to linear parabolic control problems, the optimal control problem for the Burgers equation is a non-convex problem with multiple local minima due to nonlinearity of the differential equation. Numerical methods can only compute minima close to the starting points [8].

Parabolic optimal control problems with and without constraints were solved using COMSOL Multiphysics [4, 5, 6]. In this paper, we present numerical results for distributed optimal control of unsteady Burgers equation without and with control constrains. We follow the function based " first optimize then discretize " strategy which allows to apply different optimization techniques for solving the optimality conditions.

For the approximative solution of the equations in the optimality systems, we use the classical approach of sequentially solving the state and the adjoint equations by the gradient method, interpreting the time as an additional space dimension and solving the elliptic PDE that contains the whole optimality systems by COMSOL Multiphysics like in [4, 5, 6].

We summarize first the existence and uniqueness of solutions of the unsteady Burgers equation following [8, 11]. Given $\Omega = (0,1)$ and $T > 0$, we define $Q = (0,T) \times \Omega$ and $\Sigma =$

$(0,T)\times\partial\Omega$. Let $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$ be Hilbert spaces. We make use of the following Hilbert space:

$$W(0,T) = (\varphi \in L^2(0,T;V); \varphi_t \in L^2(0,T); V^*),$$

where V^* denotes the dual space of V. The inner product in the Hilbert space V is given with the natural inner product in H as

$$(\varphi,\psi)_V = (\varphi',\psi')_H, \text{ for } \varphi,\psi \in H.$$

The expression $\varphi(t)$ stands for $\varphi(t,\cdot)$, considered as function in Ω only when t is fixed.
We consider the unsteady viscous Burgers equation

$$y_t + yy_x - \nu y_{xx} = f + bu \text{ in } Q \qquad (1)$$

with homogenous Dirichlet boundary conditions

$$y(t,0) = 0 \text{ on } \Sigma$$

and with the initial condition

$$y(0) = y_0 \text{ in } \Omega$$

where $f \in L^(Q)$ be a fixed forcing term, $\nu = \frac{1}{Re} > 0$ denotes the viscosity parameter and Re is the Reynolds number. The location and intensity of the controls $u \in L^2(Q)$ are expressed by the function $b \in L^\infty$.

For the unsteady Burgers equation (1) with the corresponding initial and boundary conditions there exists a weak solution $y \in W(0,T)$ satisfying

$$\begin{aligned} &< y_t(t),\varphi >_{V^*,V} +\nu(y_t(t),\varphi)_V \\ +\quad &(y(t)y_x(t),\varphi)_H = ((f+bu)(t),\varphi)_H \end{aligned}$$

or all $\varphi \in V$, and $t \in [0,T]$, and $(y(0),\chi)_H = (y_0,\chi)$ for all $\chi \in H$.

2 Optimal control of Burgers equations without inequality constraints

The distributed control problem for Burgers equation without inequality constraints and with homogeneous Dirichlet boundary conditions can be stated as follows [8]:

$$\min \quad J(y,u) = \frac{1}{2}\|y-y_d\|_Q^2 + \frac{\alpha}{2}\|u\|_Q^2$$

$$\begin{aligned} \text{s.t.} \quad y_t + yy_x - \nu y_{xx} &= f + bu && \text{in } Q, \\ y &= 0 && \text{on } \Sigma, \\ y(0) &= y_0 && \text{in } \Omega, \end{aligned}$$

with the regularization parameter $\alpha > 0$. Here, y and u denote the state and control variables, y_d is the desired state.

In order to show the existence of the optimal solutions, the operator $e: X \to Y$ is introduced by

$$\begin{aligned} e(y,u) &= (e_1(y,u), e_2(y,u)) \\ &= (y_t - \nu y_{xx} + yy_x - f - bu, y(0) - y_0), \end{aligned}$$

where

$$X = W(0,T)\times L^2(0,T) \text{ and } Y = L^2(0,T;V^*)\times H.$$

Then, the optimal control system above can be interpreted as a minimization problem with equality constraints

$$\text{minimize } J(y,u), \quad s.t. \quad e(y,u) = 0.$$

It was proved that there exist an optimal solution $(y^*,u^*,\lambda^*,\mu^*)$ satisfying the first-order necessary optimality conditions [1, 9, 11]

$$L'(y^*,u^*,\lambda^*,\mu^*) = 0, \quad e(y^*,u^*) = 0$$

with the augmented Lagrangian

$$\begin{aligned} L(y,u,\lambda,\mu) &= J(y,u) + (e_1(y,u),\lambda)_{L^2(V^*),L^2(V)} \\ &+ (e_2(y,u),\mu)_H. \end{aligned}$$

First-order optimality conditions lead to the following optimality system:

$$\begin{aligned} y_t^* - \nu y_{xx}^* + y^* y_x^* &= f + bu^* && \text{in } Q, \\ y^*(t,0) = y^*(t,1) &= 0 && \text{on } \Sigma, \\ y^*(0) &= y_0 && \text{in } \Omega, \end{aligned} \qquad (2)$$

$$\begin{aligned} p_t^* + \nu p_{xx}^* + y^* p_x^* &= y^* - y_d && \text{in } Q, \\ p^*(t,0) = p^*(t,1) &= 0 && \text{on } \Sigma, \\ p^*(T) &= 0 && \text{in } \Omega, \end{aligned} \qquad (3)$$

with the gradient condition

$$\alpha u^* + p^* = 0.$$

Here, u^* is the optimal control and $y*$ denotes the associated optimal state, p^* is the adjoint state.

The adjoint equation (3) can be transformed by the time transformation $\tau = T - t$ into an initial-boundary value problem

$$\begin{aligned} p_\tau^* - \nu p_{xx}^* - y^* p_x^* &= \tilde{y}_d - \tilde{y}^* && \text{in } Q, \\ p^*(\tau,0) = p^*(\tau,1) &= 0 && \text{on } \Sigma, \\ p^*(\tau = 0) &= 0 && \text{in } \Omega, \end{aligned}$$

where $\tilde{y}^*(\tau, x) = \tilde{y}^*(T - \tau, x)$.

As a numerical example we have chose the following optimal control problem in [9] with the parameters $\alpha = 0.05$, $\nu = 0.01$, $f = 0$, with the desired state $y_d(t, x) = y_0$ and with the initial condition

$$y_0 = \begin{cases} 1 & \text{in } \left(0, \frac{1}{2}\right], \\ 0 & \text{otherwise.} \end{cases}$$

The control acts on the located support $(0, T) \times (\frac{1}{4}, \frac{3}{4})$.

2.1 Sequential or iterative approach: the gradient method

Introducing the control to state operator $G : L^2(Q) \to H$ that assigns to each $u \in L^2(Q)$ the corresponding Burgers solution $y(u)$, the state variable y can be from the objective function using the solution operator G. Then, the functional $J(G(u), u)$ will be minimized by the gradient method:

$$\left(\frac{d}{du}J(G(u), u), h\right) = (G(u) - y_d, Gh) + \alpha(u, h)$$
$$= (G^*(G(u) - y_d), h) + \alpha(u, h),$$

where $h \in L^2(Q)$ is a directional vector. The descent direction is given by

$$\nu = G^*(G(u) - z) + \alpha u.$$

The adjoint state is $p := G^*(G(u) - z) = G^*(y - z)$. We use the gradient method as described in [5] where for the Burgers equation at each iteration step a nonlinear system of equation is to be solved.

After specifying the domain: $Q = (0, 1) \times (0, 1)$

```
fem.geom=solid1([0 1])
```

the solution of state equation given as:

```
fem.equ.f = { {'u-y*yx' ;0;0;0 } }
% boundary conditions
fem.bnd.r = { {'y'  0 0  0} };
fem.xmesh = meshextend(fem);
% time dependent PDE solver
fem.sol = femtime(fem,'solcomp',{'y'},...
'outcomp',{'y','p','u','uold'},...
'u',fem.sol,'tlist',[0,1]}
```

Similarly, the adjoint equation (3) is solved by redefining the boundary conditions and the coefficients for p and u. We refer to [5] for a detailed COMSOL script for building the *fem-structure* that solves time dependent PDEs in COMSOL Multiphysics.
The numerical results for different space and time meshes are listed in Table 1.

$\Delta x_{max} = \Delta t_{max}$	$\|\|J(y, u)\|\|_Q$	# iterations
2^{-3}	0.06725	32
2^{-4}	0.07233	46
2^{-5}	0.06926	73
2^{-6}	0.06778	74

Table 1: Gradient method for the unconstraint optimal control problem.

2.2 One-shot approach: treating the reverse time directions by simultaneous space-time discretization

From the gradient equation, holding in the whole space-time domain Q for the distributed control problem or the boundary Σ for the boundary control problem, we obtain $u^* = \frac{1}{\alpha}p$, where p is evaluated in the whole domain or on the boundary. When this expression is inserted into the state equation and the time variable, t is treated as an additional space variable we obtain the following boundary-value of the form:

$$\left.\begin{array}{rcl} y_t - \nu y_{xx} + y y_x &=& -\frac{1}{\alpha}p \\ p_t + \nu p_{xx} + y p_x &=& y - y_d \end{array}\right\} \text{ in } Q,$$

$$\left.\begin{array}{rcl} y &=& 0 \\ p &=& 0 \end{array}\right\} \text{ on } \Sigma,$$

$$\begin{array}{rcll} y &=& y_0 & \text{in } \Omega \times \{0\}. \\ p &=& 0 & \text{in } \Omega \times \{T\}. \end{array}$$

In the sequential approach optimality system is solved iteratively using the gradient method. The control variable u is first initialized and the state equation was solved for y forwards; the adjoint equation backwards for p until convergence. In one-shot approach, the optimality system in the whole space-time cylinder is solved as an elliptic (biharmonic) equation by interpreting the time as an additional space variable.

We used adaptive elliptic solver ***adaption***:

```
fem=adaption(fem,'ngen',2,'Maxiter',50,...
'Hnlin','off');
```

and nonadaptive elliptic solver ***femnlin***

```
fem.sol=femnlin(fem);
```

We have used quadratic finite elements for the state y and adjoint variable p.

Figure 1 shows the computed optimal control u_h, the computed optimal state y_h and the associated adjoint state u_h for the one-shot approach with adaptation for $h = \Delta x_{max} = 2^{-6}$. The numerical solutions obtained by the gradient method and by the one-shot approach without adaptation are similar to those in Figure 1. The adaptive mesh for $h = \Delta x_{\max} = 2^{-4}$ is given in Figure 2.

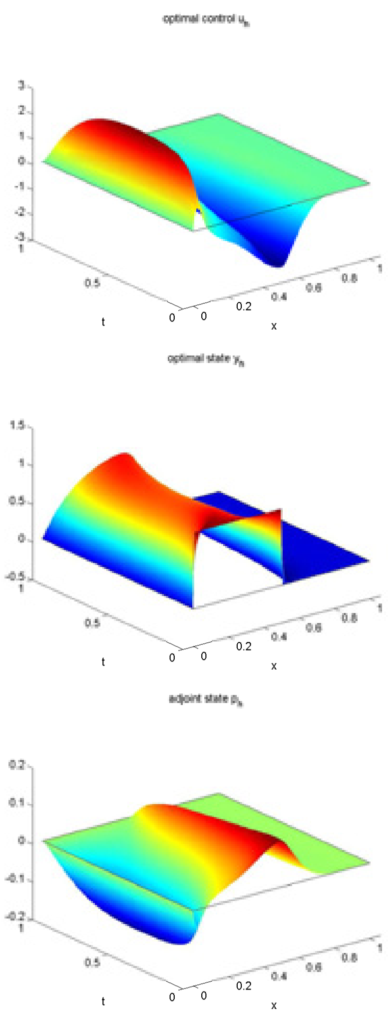

Figure 1. One-shot approach with adaptation for the unconstrained problem.

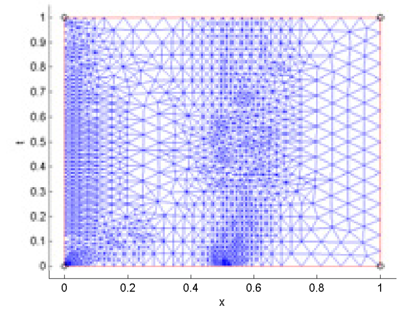

Figure 2. Adaptive mesh of the one-shot approach for the unconstrained problem.

The numerical results for different mesh sizes are given in Table 2.

Δx_{max}	$\|\|J(y,u)\|\|_Q$ with adaption	$\|\|J(y,u)\|\|_Q$ with femnlin
2^{-3}	0.0663	0.0651
2^{-4}	0.0667	0.0686
2^{-5}	0.0667	0.0671
2^{-6}	0.0667	0.0669

Table 2: One-shot approach for the unconstraint control problem.

3 Optimal control of Burgers equation with inequality control constraints

We consider now distributed optimal control problem with bilateral control constraints [8]

$$\min \quad J(y,u) = \frac{1}{2}\left\|y - z\right\|_Q^2 \; + \frac{\alpha}{2}\left\|u\right\|_Q^2$$

$$\begin{aligned} \text{s.t.} y_t + yy_x - \nu y_{xx} &= f + bu \text{ in } Q, \\ y &= 0 \text{ in } \Sigma, \\ y(,\cdot) &= y_0 \text{ in } \Omega, \end{aligned}$$

with pointwise control constraints

$$u_a(t,x) \le u(t,x) \le u_b(t,x) \text{ in } Q$$

First-order necessary optimality conditions for the local solution (y^*, u^*) have to be satisfied with the adjoint variable p^* in form of the optimality system (2) and (3) including the control constraints $u^* \in U_{ad} = \{u \in L_2(Q) : u_a(t,x) \le u(t,x) \le u_b(t,x)\}$. Because of the pointwise constraints, additionally we have the variational inequality

$$\int_Q (\alpha u^* + bp^*)(u - u^*)dxdt \ge 0 \text{ for all } u \in U_{ad}.$$

The last inequality can be expressed in form of the projection:

$$u^*(t,x) = P_{[u_a(t,x),u_b(t,x)]}\left(\frac{b(t,x)}{\alpha}p^*(t,x)\right)$$

As a numerical example we consider the unilaterally control constrained bounded problem ($u \le u_b$) with the initial condition $y_0 = \sin(13x)$, $\nu = 0.1$, $u_b = 0.3$ and regularization parameter $\alpha = 0.01$ in [7]. The desired state is taken as the initial condition $y_d = y_0$

We have used the projection method as in [4] to implement complementary slackness conditions,

$$(\mu, u - b) = 0, u \le b, \mu \ge 0.$$

The projection method handles the complementary slackness conditions by replacing this conditions by a projection. This is an implementation of the active set strategy as a semi smooth Newton method [2]. It can be shown that complementary slackness conditions are equivalent to

$$\mu = c\max(0, \frac{\mu}{c} + u - b) \;\; \text{for any } c > 0.$$

By choosing $c = \alpha$ and eliminating μ from the gradient equation $\alpha u + p + \mu = 0$ we get

$$\mu = \alpha\max(0, -b - \frac{p}{\alpha}) \text{ a.e. in } Q.$$

In COMSOL Multiphysics, the projection method is implemented in the following form:

- using the one shot approach

```
fem.equ.f= { {'-ytime-(p+mu)/alpha-yyx'
'ptime+y-zd(x,time)+y*px'...
'(1/alpha)*mu-max(0,-b-(1/alpha)*p)'}};
```

- using the gradient method

```
fem.equ.f ={ {0;0;0;0;'(1/alpha)*mu
-max(0,-b-(1/alpha)*p)' } };
```

We then define the *fem* structure, to solve the optimality system by one call of nonlinear solver ***femnlin***. This solver is an affine invariant form of the damped Newton method. This solver is often used to solve problems with the augmented Lagrangian technique.

Numerical results of the gradient method are given in Table 3:

$\Delta x_{max} = \Delta t_{max}$	$\|\|J(y,u)\|\|_Q$	# of iterations
2^{-3}	0.2155	52
2^{-4}	0.20824	65
2^{-5}	0.2023	219
2^{-6}	0.2006	524

Table 3: Gradient method for the control constraint problem.

We have solved the control constraint using the one-shot approach with the adaptive solver **adaption** and with without adaptation using **femnlin**. The ***fem*** structure in COMSOL Multiphysics contains the geometry of the domain, the coefficients of the PDEs, etc. The following lines are from the one-shot approach:

```
fem.form='general'; fem.globalexpr= {'u'...
 '-(p+mu)/alpha' };
fem.equ.ga= { { {'-nu*yx' '0'} {'-nu*px '...
 '0' }... {'0' '0' } }};
fem.equ.f= { {'-ytime-(p+mu)/alpha-y*yx'...
 'ptime+y- ...zd(x,time)+y*px' ...
 '(1/alpha)*mu-max(0,-0.3-(1/alpha)*p)' } };
fem.bnd.ind=[1 2 3 2];
%Boundary conditions
fem.bnd.r= { {'y-y0(x)' 0 0 };...
 {'y' 'p' 0 }; {0 'p' 0 } };
fem.bnd.g= { {0 0 0 }; {0 0 0 };...
 {0 0 0 } };
% Postprocessing
postplot(fem,'tridata','y','triz','y')
```

We used for state and adjoint state variables quadratic finite elements like in the unconstraint case, for the Lagrange multiplier μ, linear finite elements are taken.

Δx_{max}	$\|\|J(y,u)\|\|_Q$ with adaption	$\|\|J(y,u)\|\|_Q$ with femnlin
2^{-3}	0.2000	0.1985
2^{-4}	0.2002	0.2000
2^{-5}	0.2003	0.2002
2^{-6}	0.2003	0.2003

Table 4: One-shot approach for the control constraint problem .

In Figure 3, the computed solutions are given for the control constraint problem for $\Delta x_{max} = 2^{-6}$. The numerical solutions obtained by the gradient method and by the one-shot approach without adaptation are similar to those in Figure 3. The adaptive mesh for $h = \Delta x_{\max} = 2^{-4}$ is given in Figure 4.

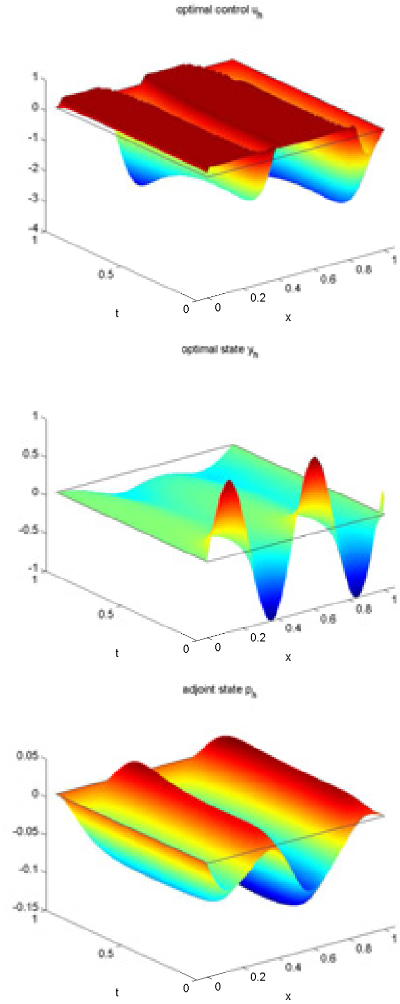

Figure 3. One-shot approach with adaptation for the control constrained problem.

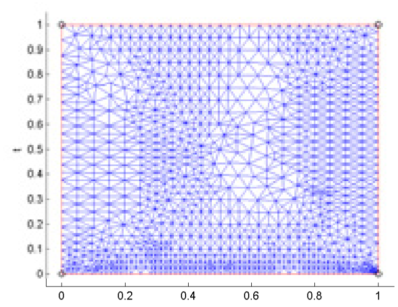

Figure 4. Adaptive mesh of the one-shot approach for the control constrained problem.

4 Conclusion

We have shown that the finite element package of COMSOL Multiphysics can be used for solving time-dependent non-linear optimal control problems. For the viscous Burgers equation it was shown that when the optimality conditions are available in for of PDEs, the specialized finite elements solvers can be easily implementable. Both classical gradient based approach solving the state equation forward in time and the adjoint equation backward in time and solving the the whole optimality system as an biharmonic equation produces satisfactory results for the Burgers equation.

The applicability of this approach should be tested for Burgers equation with state constraints as it was done in [6] for parabolic control problems. We also plan to apply the various stabilization techniques available in COMSOL Multiphysics to the Burgers equation.

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